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When the Landau criterion fails qualitatively

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Abstract. We show that a theory with an order parameter of one real component, ϕ, with $u_3\phi^3$ local interaction exhibits—due to fluctuations—a first order phase transition for $|u_3| > u_{3c} > 0$, a critical point at $u_3 = u_{3c}$, and no transition for $|u_3| < u_{3c}$. $u_3 = 0$ is a special point of second order phase transition even for $u_{3c} > 0$. This is in contradiction to the Landau theory which predicts a first order transition for all $u_3 > 0$. The relation of this failure of Landau's theory to questions of symmetry breaking is discussed.

1. Introduction: the Landau theory, symmetry first order transitions and fluctuations

It is part of the lore of the phase transition community that while the simple Landau theory is quantitatively wrong, it provides an adequate qualitative description. In particular, it is believed that the order of a phase transition is correctly described by the simple Landau theory. An even stronger belief is that above four dimensions the Landau description should also become quantitatively correct. Here we will study a system with a ϕ^3 interaction in some detail. It will serve to show that when the Landau theory predicts a first order phase transition, and no symmetry is broken in the transition, this theory is not reliable.

The existence of a cubic term in the expansion of the free energy in the order parameter plays a crucial role in the Landau argument. When such a term is allowed, a first order transition is predicted. When fluctuations are taken into account the nature of the transition still depends on the actual values of the bare parameters. As in the case of a liquid–gas transition, there is a line of first-order transitions, a critical point and a transitionless regime. The Landau theory predicts the transition temperature correctly when one occurs. It fails in detecting the transitionless regime and in predicting the nature of the transition.

An important aspect which is, we believe, emphasized by the present argument, is the role of symmetry breaking in determining the nature of phase transitions. Traditionally the symmetry is invoked in the derivation of the Landau expansion of the free energy (Landau and Lifshitz 1968) but actually plays no role at any subsequent stage. In particular the procedure predicts a first order transition whenever there is a cubic term in the expansion of the free energy. The situation in the Wilson theory is quite different. The Hamiltonian is postulated and symmetry has been used explicitly in the renormalization procedure (Wegner 1972, Wallace and Zia 1975, Zia and Wallace 1975). Actually it plays an essential role in determining the nature of the transition. As we will show explicitly, the Landau theory is misleading when no symmetry is broken. On the other hand the predictions of the order of the transition seem to be correct when a symmetry is broken (and $d > 2$). In such situations the Landau theory predicts the order of the transition but does not predict the transition temperature correctly.

Probably the simplest, and certainly the most well studied, case is the spin 1 Potts model and its continuous modifications. It has been shown (Alexander and Yuval 1974, Alexander 1974, 1975) that this model describes a cubic to tetragonal transition in spin space. Explicit series expansions of this model were discussed by a number of authors (Potts 1952, Kihara *et al* 1954, Mittag *et al* 1971, Straley and Fisher 1973, Alexander and Yuval 1974, Enting 1974) and the two-dimensional model was shown by Baxter (1973) to have a continuous transition which seems to be confirmed experimentally (Alexander 1975). The continuous three-dimensional model was treated by Golner (1973) using explicit renormalization group techniques, and more recently it was shown in an ϵ-expansion (Amit and Scherbakov 1974, Wallace and Zia 1975, Zia and Wallace 1975) that the cubic term is relevant near the ($n = 2$) fixed point. It is thus almost certain that the transition is of first order. This seems to be true in spite of the fact that fluctuations can be very important near the transition as shown by Alexander (1974).

The situation is quite different when no symmetry is broken. A number of authors (Griffiths 1967, Harris 1968, Priest 1971, Shultz 1971) have considered what is actually the analogue of the Potts model when no symmetry is broken. Their model is a spin 1 model with biquadratic axially symmetric interactions. The connection with the Potts model and the fact that the two models have identical mean field theories was pointed out by Alexander and Yuval (1974). This model leads directly to a Hamiltonian of the type we will discuss below. It can, however, also be mapped on an Ising model, with a temperature dependent field, and in particular Shultz has used this fact to study the phase diagram in detail. He finds that the crucial parameter is the suppression of the transition temperature of the relevant Ising model compared with the mean field result, which in this case depends only on the range of the spin–spin interaction. Shultz finds that there is no transition for nearest neighbour interactions, and the critical point occurs when the range is about a thousand lattice spacings. Obviously this implies that fluctuations are extremely important at the mean field (Landau) transition temperature for nearest neighbour interactions and as a result the transition disappears.

Since the Potts model has the same mean field theory and similar fluctuations, these must certainly be very important near the mean field transition temperature. This is also indicated by a calculation of a Ginsburg criterion (Benguigui 1974). It does in fact follow from the argument of Alexander (1974) that no transition can occur at this temperature. Since the symmetry change precludes the possibility of a critical point, it follows that the transition temperature is suppressed, and that there is a temperature range with important fluctuations, in spite of the fact that the transition is of first order.

2. The model

We consider a system described by a real order parameter field $\phi(\boldsymbol{x})$. The Hamiltonian density giving the statistical weight of a given distribution $\phi(\boldsymbol{x})$ is

$$\mathscr{H}(\boldsymbol{x}) = \tfrac{1}{2}(\nabla\phi)^2 + \tfrac{1}{2}u_2\phi^2 + u_3\phi^3 + u_4\phi^4. \tag{1}$$

The partition function, averages and correlation functions are calculated with the weight (Ferrell 1969, Wilson 1971, Amit and Zannetti 1973)

$$W\{\phi\} = \exp\left(-\int \mathscr{H}(\boldsymbol{x})\,\mathrm{d}x\right). \tag{2}$$

This model is rather appropriate for the description of the gas–liquid transition, a fact that is obscured by the Landau theory.

The Landau theory (or the mean field theory) is obtained in this formalism by searching for the ϕ which maximizes W, or minimizes $\int \mathscr{H}$ (Siegert 1963, Amit and Zannetti 1973).

In the bulk system, it is a uniform $\phi = \bar{\phi}$ which minimizes the exponent, and the Landau equation is

$$\bar{\phi}(u_2 + 3u_3\bar{\phi} + 4u_4\bar{\phi}^2) = 0 \tag{3}$$

which corresponds to a free energy

$$F = \tfrac{1}{2}u_2\phi^2 + u_3\phi^3 + u_4\phi^4. \tag{4}$$

Following the canonical procedures one finds that, for any given u_3 and u_4, there is a first order phase transition at

$$u_2 = \tfrac{1}{2}u_3^2/u_4 \equiv u_2^{\mathrm{L}}. \tag{5}$$

One thus notes that u_3 introduces a new scale of temperature (or length) into the problem. For this value of u_2 the equilibrium value of $\bar{\phi}$ changes abruptly from zero to $\bar{\phi} = -\frac{1}{2}u_3/u_4$—a finite jump if $u_3 \neq 0$.

Graphically the situation is depicted in figure 1. For $u_2 > u_2^{\mathrm{L}}$ and $u_3 > 0$ one obtains a situation described by curves A or B. At $u_2 = u_2^{\mathrm{L}}$ the situation is given by curve C, and finally when $u_2 < u_2^{\mathrm{L}}$, the minimum at $\bar{\phi} \neq 0$ is the absolute equilibrium.

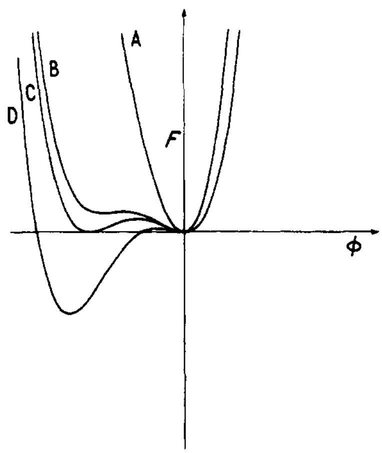

Figure 1. The Landau free energy function against the order parameter for various values of u_2. In curves A and B $u_2 > u_2^{\mathrm{L}}$, in curve C $u_2 = u_2^{\mathrm{L}}$ and in curve D $u_2 < u_2^{\mathrm{L}}$.

The susceptibility is continuous across the phase transition line and its value is

$$\chi^{-1} = u_2^{\mathrm{L}} = \tfrac{1}{2}u_3^2/u_4 \tag{6}$$

and the discontinuity of the entropy is given by

$$\Delta S = \left.\frac{\partial F}{\partial T}\right|_{\phi=\bar{\phi}} - \left(\frac{\partial F}{\partial T}\right)_{\phi=0} = \tfrac{1}{8}(u_3^2/u_4^2)\left.\frac{\partial u_2}{\partial T}\right|_{u_2=u_2^{\mathrm{L}}}. \tag{7}$$

3. The real behaviour of the system

We consider again the system described by equation (1) in the absence of an external field. The model under consideration is simple enough, so one can make completely general statements concerning its phase transition based on the known properties of the Ising model (Wilson and Kogut 1974, Brezin *et al* 1974b)—the model with $u_3 = 0$. As was shown by Brezin *et al* (1974a), no new anomalous dimensions are introduced by the composite operator ϕ^3, since it is connected by an equation of motion to ϕ. But this does not reveal much concerning the nature of the transition. We write

$$\phi = \psi + M \tag{8}$$

where M is a constant which we choose as

$$M = -u_3/4u_4 \tag{9}$$

in order to eliminate the cubic term. In terms of the variable ψ, $\mathscr{H}(x)$ reads as

$$\mathscr{H}(x) = (u_3/u_4)^2(\tfrac{1}{2}u_2 - \tfrac{3}{16}u_3^2/u_4) + \tfrac{1}{2}(\nabla\psi)^2 - (u_3/4u_4)(u_2 - \tfrac{1}{2}u_3^2/u_4)\psi + (\tfrac{1}{2}u_2 - \tfrac{1}{2}u_3^2/u_4)\psi^2 + u_4\psi^4. \tag{10}$$

The problem is transformed into a usual ψ^4 interaction, in the presence of an effective external field, which depends on u_2 and u_3:

$$h = (u_3/4u_4)(u_2 - \tfrac{1}{2}u_3^2/u_4). \tag{11}$$

As is well known, there will be a phase transition only if the field h vanishes. That is, if either

$$u_3 = 0 \tag{12}$$

or

$$u_2 = \tfrac{1}{2}u_3^2/u_4. \tag{13}$$

The first case is, of course, the usual Ising model. It is special, in the present context, since u_2 remains free, and

$$u_2 = u_{2c} < 0 \tag{14}$$

is a point of second order phase transition (Wilson 1972). u_{2c} is the temperature of the transition relative to the mean field temperature—a depression caused by fluctuations. If $u_2 < u_{2c}$ one is on the co-existence curve (Brezin *et al* 1972, 1973), while if $u_2 > u_{2c}$ all quantities are regular functions of u_2—there is no transition.

The second case, equation (13), is more interesting. The external field is still kept zero, and u_2 from equation (13) is substituted into equation (10) to yield

$$\mathscr{H}(x) = (u_3/4u_4)^2(u_3^2/16u_4) + \tfrac{1}{2}(\nabla\psi) - (u_3^2/4u_4)\psi^2 + u_4\psi^4. \tag{15}$$

Once again, using the known results of the ψ^4 theory, we conclude that the Hamiltonian of equation (15) describes a regular phase if

$$u_3^2/4u_4 < |u_{2c}|, \tag{16}$$

a critical point if

$$u_3^2/4u_4 = |u_{2c}| \tag{17}$$

and a coexistence curve if

$$u_3^2/4u_4 > |u_{2c}|. \tag{18}$$

With the Hamiltonian we have assumed in equation (1), the value of u_{2c} depends only on u_4 and on the cutoff Λ—the inverse of the lattice spacing.

The situation is described in figure 2 which is a plane of zero external field, h_e, and fixed u_4. For a given u_4 and Λ, if u_2—the temperature—is varied at constant u_3 then if u_3 satisfies equation (16) the path will be the one denoted by A in figure 2; there will be regular behaviour—ie no transition; if u_3 satisfies equation (17), the system will follow path B, on which there will be a second order phase transition—a critical point. Finally if equation (18) is satisfied by u_3, path C will be followed and a first order phase transition will show up as the line $u_2 = \frac{1}{2}u_3^2/u_4$ is crossed.

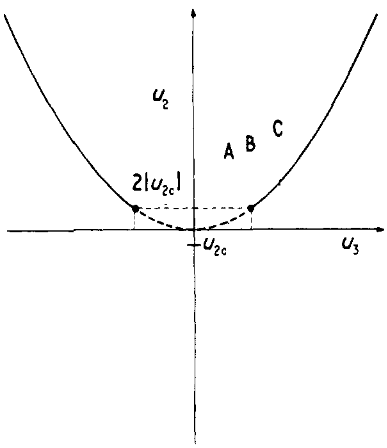

Figure 2. The phase diagram of the system for zero external field and given u_4. The broken part of the zero effective field parabola is the part across which the Landau theory predicts a first order transition, but in fact there is none.

4. Discussion of the phase transition

In fact, the above discussion establishes the result mentioned in § 1 for the case when no external field is applied. A few comments are, however, in place.

(i) The phase diagram is, as was mentioned in § 1, analogous to the gas–liquid phase diagram, if one keeps to u_3 of a given sign. The diagram is symmetric in $u_3 \to -u_3$, since this replacement amounts to a change $\phi \to -\phi$.

(ii) As one moves along one of the lines A, B, C, in figure 2, not only u_2—the temperature—is varied, but also the effective field h, equation (11). This field vanishes along the whole length of the parabola given by equation (13). On crossing this curve, h changes sign. This is what brings about a first order transition when the zero field curve is crossed along C. On the other hand the critical behaviour along B is a combination of temperature and effective field effects. This we shall see in more detail below.

(iii) If one proceeds along the parabola $h = 0$ from the origin until the point

$$u_3^2/4u_4 = |u_{2c}|$$

is reached, the equilibrium value of ψ is zero. Beyond this point, further up the parabola, ψ has a non-zero value (Brezin *et al* 1972). This, however, does not imply that a symmetry is broken, since from equations (8) and (9) it follows that, for $u_3 \neq 0$, the equilibrium value of ϕ is non-zero on both sides of the transition. In fact, when $u_3 \neq 0$, the equilibrium value of ϕ is never zero, which is a point we shall return to. This is in contrast to the Landau theory (see eg § 1), which predicts a symmetry breakdown, and hence cannot allow for a regular path going from one phase to the other. But the Hamiltonian, equation (1), possesses no symmetry to be broken.

5. In the presence of an external field

The introduction of an external field changes $\mathscr{H}(x)$, equation (10), in two respects. First, the constant term—independent of ψ—becomes

$$h_e(u_3/4u_4)+(u_3/4u_4)^2(\tfrac{1}{2}u_2-\tfrac{3}{16}u_3^2/u_4)$$

and the effective field becomes

$$h = h_e+(u_3/4u_4)(u_2-\tfrac{1}{2}u_3^2/u_4). \tag{19}$$

Now, no transition can occur for $u_3 = 0$. The condition for the vanishing of the effective field, which replaces equation (13), is

$$u_2 = \tfrac{1}{2}u_3^2/u_4-(4u_4/u_3)h_e. \tag{20}$$

When this condition is satisfied, one finds for the coefficient of ψ^2 in equation (10):

$$v_2 = -\tfrac{1}{4}u_3^2/u_4-(2u_4/u_3)h_e. \tag{21}$$

In a plane of a given h_e, the two symmetrically placed transition lines of figure 2 are replaced by the lines given by equation (20) and the condition

$$-v_2 > |u_{2c}|. \tag{22}$$

Two typical situations can arise; they are depicted in figure 3. The first one—the broken curve—is the case of small external field h_e. In this plane there are still three second order points C_+, C_- and C_0. To lowest order in h_e these points are given by

$$\begin{aligned} u_{3\pm} &= \pm(4u_4|u_{2c}|)^{1/2}-(u_4|u_{2c}|)h_e \\ u_{2\pm} &= 2|u_{2c}|\pm 8u_4(4u_4|u_{2c}|)^{-1/2}h_e \end{aligned} \tag{23}$$

and the points which move away from the Ising critical point are

$$u_{3_0} = (4u_4/|u_{2c}|)h_e \qquad u_{2_0} = u_{2c}+(16u_4/u_{2c}^2)h_e^2. \tag{24}$$

As the field h_e increases, the critical points C_+ and C_0 merge, and there is no second order transition for $u_3 > 0$. Only one such point remains; it is marked D_-. The curve of first order transitions is the broken curve in figure 3. For comparison the parabola in $h = 0$ is drawn as a full curve.

An illuminating way of viewing the locus of phase transition points geometrically was suggested to us by David Wallace. In three dimensional space (h_e, u_2, u_3), there are one-parameter families of Hamiltonians which give rise to identical physics. Each family is generated from any given $\mathscr{H}$ by translating ϕ by a constant M, $-\infty \leqslant M < \infty$. The point $(0, u_{2c}, 0)$ is a point of second order transition of the Ising type. The family

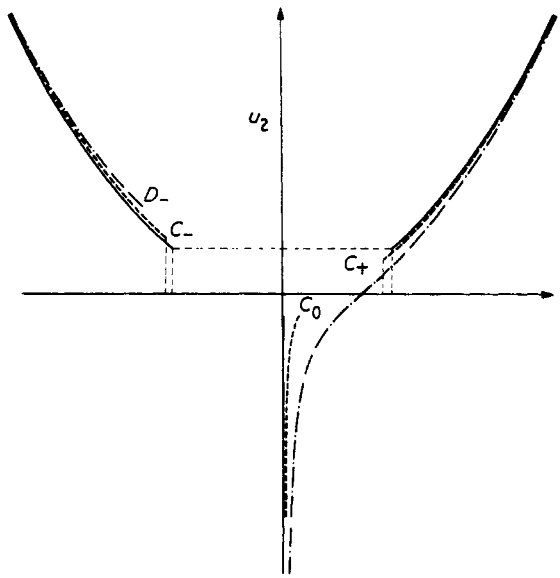

Figure 3. The phase diagram in a plane of non-zero external field.

generated from it is $(u_{2c}M+4u_4M^3, u_{2c}+12u_4M^2, 4u_4M)$, which is a curve, $C^{(2)}$, of second order transition points in the three dimensional space. This curve crosses the $h_e = 0$ plane at three points

$$M = 0, \pm(-u_{2c}/4u_4)^{1/2}$$

which are the three points of second order transition in figure 2. It also crosses the plane of small h_e at three points—C_+, C_-, C_0 in figure 3, and for large h_e only at one point D_- in figure 3.

The point $(0, u_2, 0)$ for $u_2 < u_{2c}$ represents a Hamiltonian which describes a point of first order transition as h_e passes through zero. Thus the curve, $C^{(2)}$, described in the previous paragraph is the edge of a surface of first order transition points. The surface is made of straight lines starting on $C^{(2)}$ parallel to the plane $M = u_3 = 0$. The direction of the line in the (h_e, u_2) plane, for a given value of M (or u_3), is

$$\tan\sigma = M^{-1}.$$

As $M \rightarrow -\infty$ this line is perpendicular to the (u_2, u_3) plane in the direction of negative h_e. It coincides with the line $h_e = 0$, $u_3 = 0$ for $M = 0$, $u_2 < u_{2c}$. This is the line of coexistence points of the Ising model. Finally as $M \rightarrow \infty$ the line becomes perpendicular to the $h_e = 0$ plane in the direction of $+h_e$.

6. A digression on the role of symmetry

It was mentioned above that the failure of the Landau theory in the present case is related to the introduction of spurious symmetry on the zero-loop level (Coleman and Weinberg 1973, Brezin *et al* 1974b). Once fluctuations are considered, which *inter alia* bring about the depression of the transition temperature $u_{2c} \neq 0$, one finds 'tadpole' graphs in the free energy. Namely, terms proportional to the first power of the equilibrium value of the order parameter. At the one-loop level one finds the following term:

$$u_3\bar{\phi}D_1(m)$$

where

$$D_1(m) = \int \frac{\mathrm{d}q}{(2\pi)^d}\left(\frac{1}{q^2+m^2}\right).$$

This term is generated by the ϕ^3 coupling in the Hamiltonian, equation (1). In equation (3) there will be terms independent of $\bar{\phi}$, and $\bar{\phi} = 0$ will not be a solution at any temperature.

This situation should be contrasted with the continuous Domb–Potts model (Potts 1952a, b, Golner 1973, Amit and Scherbakov 1974). There, the cubic coupling is of the form $u\,(\phi_1^3 - 3\phi_1\phi_2)$. Such a term still has a three fold symmetry of discrete rotations, by 120°, in the ϕ_1–ϕ_2 plane. It can be shown that no 'tadpoles' can appear in the free energy, so that the high temperature solution remains exactly at the origin, and there is a transition for all values of u_3. The order of this transition is predicted correctly by the Landau theory. It is probably always first order.

7. The critical region in the ϵ-expansion

In order to see the structure of the theory in the neighbourhood of the critical point, we resort to the ϵ-expansion, and calculate to first order in $\epsilon = 4-d$, where d is the number of space dimensions. The calculation proceeds along lines which are by now very familiar, and so we restrict ourselves to a very brief account. The ideas are those of Wilson (1972) and Brezin *et al* (1972, 1973). The particular form used here follows Amit and Scherbakov (1973) and Amit (1974).

The appearance of a non-zero equilibrium value for ϕ is taken care of by a shift

$$\phi(x) = \psi(x) + M$$

with

$$\int \psi(x)\,\mathrm{d}x = 0 \tag{25}$$

and M is determined by minimizing the free energy rather than by equation (9). This is equivalent to an integration over M, or to the relaxation of equation (25) and the imposition of the constraint $\langle\psi\rangle = 0$ (Brezin *et al* 1973).

$$F(M) = F_{\mathrm{L}}(M) + \Delta F(M) \tag{26}$$

where

$$F_{\mathrm{L}}(M) = \tfrac{1}{2}u_2M^2 + u_3M^3 + u_4M^4 \tag{27}$$

is the tree (zero-loop) or Landau approximation.

$$\Delta F(M) = -\Omega^{-1}\ln\int \mathscr{D}\psi\,\exp\left(-\int H'(\psi, M)\right)\mathrm{d}x \tag{28}$$

with

$$H'(\psi, M) = \tfrac{1}{2}(\nabla\psi)^2 + \tfrac{1}{2}(u_2 + 6u_3 M + 12u_4 M^2)\psi^2 + (u_3 + 4Mu_4)\psi^3 + u_4\psi_4. \tag{29}$$

The term linear in ψ vanishes when integrated over the volume because of equation (25)

Next, following Brezin *et al* (1972), we renormzlize the mass, and write

$$H'(\psi, M) = [\tfrac{1}{2}(\nabla\psi)^2 + \tfrac{1}{2}m^2\psi^2] + H_{\text{int}} \tag{30}$$

$$H_{\text{int}} = \tfrac{1}{2}\,\delta m^2\psi^2 + (u_3 + 4Mu_4)\psi^3 + u_4\psi^4$$

$$\delta m^2 = u_2 + 6u_3 M + 12u_4 M^2 - m^2. \tag{31}$$

m^2 is chosen as the full inverse susceptibility.

If we choose $u_4 \sim \epsilon$, $u_3 \lesssim \epsilon^{1/2}$, $M \sim \epsilon^{-1/2}$ then the equation for $\partial F/\partial M$ to order $\epsilon^{1/2}$ is

$$u_2 M + 3u_3 M^2 + 4u_4 M^3 + (3u_3 + 12u_4 M)D_1(m) = 0 \tag{32}$$

where the mass renormalization equation

$$\delta m^2 + 12u_4 D_1(m) - 18(u_3 + 4u_4 M)^2 D_2(m) = 0 \tag{33}$$

(calculated to the same order) was utilized in deriving equation (32). D_1 and D_2 are defined as

$$D_1(m) = \int [\mathrm{d}q/(2\pi)^d](q^2 + m^2)^{-1} \tag{34}$$

$$D_2(m) = \int [\mathrm{d}q/(2\pi)^d](q^2 + m^2)^{-2}. \tag{35}$$

Since $D_2(m) \to \infty$ when $m \to 0$, for $\epsilon > 0$, if there is to be a critical point, ie a solution with $m = 0$, we must have

$$M = -u_3/4u_4 \tag{36}$$

which is identical to equation (9). This can happen only for a special combination of the parameters. Namely

$$(u_3/4u_4)(u_2 - \tfrac{1}{2}u_3^2/u_4) = 0 \tag{37}$$

which follows from equation (32), and from equation (33) it follows that at this point

$$u_2 - \tfrac{3}{4}u_3^2/u_4 = -12u_4 D_1(0) = u_{2\mathrm{c}} \tag{38}$$

to order ϵ. These equations reproduce equations (13) and (17) to first order in ϵ, if $u_3 \neq 0$. The other solution is $u_3 = 0$ and $u_2 = u_{2\mathrm{c}}$.

Away from the critical point, for general values of m, we expand about $M = -u_3/4u_4$:

$$M = -u_3/4u_4 + \mu. \tag{39}$$

Equations (33) and (32) become, respectively,

$$m^2 = (u_2 - \tfrac{3}{4}u_3^2/u_4) + 12u_4\mu^2 + 12u_4 D_1(m) - 288u_4^2\mu^2 D_2(m) \tag{40}$$

and

$$4u_4\mu^3 + (u_2 - \tfrac{3}{4}u_3^2/u_4)\mu - (u_3/4u_4)(u_2 - \tfrac{1}{2}u_3^2/u_4) + 12u_4\mu D_1(m) = 0. \tag{41}$$

The critical point is characterized by $\mu = m = 0$, and thus

$$u_{2c} - \tfrac{3}{4}u_{3c}^2/u_4 = -12u_4 D_1(0). \tag{42}$$

Defining

$$t = u_2 - u_{2c} \tag{43}$$

$$p = (u_3^2 - u_{3c}^2)/u_4 \tag{44}$$

we can rewrite equations (40) and (41) as

$$m^2 = t - \tfrac{3}{4}p + 12u_4\mu^2 + 12u_4\Delta D_1(m) - 288u_4^2\mu^2 D_2(m) \tag{45}$$

$$4u_4\mu^3 + [t - \tfrac{3}{4}p + 12u_4\Delta D_1(m)]\mu - (t - \tfrac{1}{2}p)(u_3/4u_4) = 0. \tag{46}$$

To lowest order in ϵ

$$\Delta D_1 = D_1(m) - D_1(0) \simeq Sm^2 \ln(m/\Lambda) \tag{47}$$

$$D_2(m) \simeq -S[\tfrac{1}{2} + \ln(m/\Lambda)]. \tag{48}$$

The parabola of zero effective field in figure 2 can be also described by

$$p = 2t. \tag{49}$$

Along this curve equations (45) and (46) take on the form

$$m^2 = -\tfrac{1}{2}t + 12u_4\mu^2 + 12u_4 Sm^2 \ln(m/\Lambda) + 288u_4^2 S\mu^2[\tfrac{1}{2} + \ln(m/\Lambda)] \tag{50}$$

$$4u_4\mu^3 - [\tfrac{1}{2}t - 12u_4 Sm^2 \ln(m/\Lambda)]\mu = 0. \tag{51}$$

The consistency of the orders in ϵ in the above equations is secured by the fact that, to first order in ϵ,

$$12u_4 S = \tfrac{1}{3}\epsilon$$

and

$$u_4\mu^2 = \mathrm{O}(1)$$

(see Brezin *et al* 1972, 1973).

Equation (51) has a solution with $\mu = 0$. It then follows from equation (50) that we must have $t < 0$, ie below the critical point in figure 2. The behaviour of m, as a function of t, is just like in the Ising model (see eg Wilson 1972). Next we ask whether there may may be a solution of equations (50) and (51) with $\mu \neq 0$. Substituting μ^2 from equation (51) in equation (50), one finds

$$m^2 = t - \tfrac{2}{3}\epsilon m^2 \ln(m/\Lambda) + 2\epsilon[\tfrac{1}{2}t - \tfrac{1}{3}\epsilon m^2 \ln(m/\Lambda)][\tfrac{1}{2} + \ln(m/\Lambda)]. \tag{52}$$

If terms of zeroth order in ϵ are compared one obtains

$$m^2 = t + \mathrm{O}(\epsilon) \tag{53}$$

which has no solution for $t < 0$. Thus, along the low part of the zero field parabola there is a single regular solution, $\mu = 0$.

As we have seen, for $t > 0$ there is no solution with $\mu = 0$. The system is described again by the solutions of equation (52), and hence of equation (53). There are now two solutions, which to lowest order in ϵ are given by the two roots of equation (51). Along the parabola of zero field both F_L and δm^2 are even functions of μ, and thus the free energy for the two solutions is the same. This is a line of first order transitions.

8. Conclusion

We have shown that a model with a real scalar order parameter, which has a $u_3\phi^3$ interaction in addition to $u_4\phi^4$, has a critical value of $u_3 = u_{3c}(u_4, \Lambda)$ below which there is no phase transition. At the critical value the system undergoes a second order transition with no symmetry break. Above that critical value of u_3 the transition is of first order. This is contrary to the prediction of Landau's theory, as applied to the same model, which is that there is a first order transition for all $u_3 \neq 0$.

The result holds also for $d > 4$, since it depends only on the fact that $u_{2c} \neq 0$. Namely, there is a finite depression of the transition temperature from its mean-field value. What the Wilson theory tells us, as can be seen from § 6, is that above four dimensions the exponents at the critical point will be Gaussian.

Since the value of u_{2c} is not a universal quantity, one may ask about the effects of higher powers of the field in the Hamiltonian. Such effects would not invalidate the results of the present work, since these results are all stated within a specified model. It seems, however, that the higher powers would not change the qualitative results reported here. The reason is that near the critical points they are irrelevant, and thus the two second order phase transition points will survive. Their persistence will preserve the structure of the u_2, u_3 phase diagram, figure 2. But this question has not been investigated in detail.

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References

Alexander S 1974 *Solid St. Commun.* **14** 1069
—— 1975 to be published
Alexander S and Yuval G 1974 *J. Phys.* C: *Solid St. Phys.* **7** 1609
Amit D J 1974 *J. Phys.* C: *Solid St. Phys.* **7** 3369
Amit D J and Shcherbakov A 1973 *J. Statist. Phys.* **9** 317
—— 1974 *J. Phys.* C: *Solid St. Phys.* **7** 196
Amit D J and Zannetti M 1973 *J. Statist. Phys.* **7** 31
Baxter R J 1973 *J. Phys.* C: *Solid St. Phys.* **6** L445
Benguigui L 1975 *J. Phys.* C: *Solid St. Phys.* **8** 17
Brezin E, Wallace D J and Wilson K G 1972 *Phys. Rev. Lett.* **29** 591
—— 1973 *Phys. Rev.* B **7** 232
Brezin E, De Domincis C T and Zinn-Justin J 1974a *Lett. Nuovo Cim.* **9** 483
Brezin E, Le Guillou J C and Zinn-Justin J 1974b *Phase Transitions and Critical Phenomena*, vol 6 eds C Domb and M Green (New York: Academic Press)
Coleman S and Weinberg E 1973 *Phys. Rev.* D **7** 1888
Enting I G 1974 *J. Phys.* A: *Math., Nucl. Gen.* **7** 2181
Ferrell R A 1969 *Contemporary Physics* vol I (Vienna: IAEA)
Golner G R 1973 *Phys. Rev.* B **8** 3419
Griffiths R B 1967 *Physica* **33** 689
Harris A B 1968 *Solid St. Commun.* **6** 1949
Kihara T, Midzuo Y and Shizuma T 1954 *J. Phys. Soc. Japan* **9** 681
Landau L D and Lifshitz E M 1968 *Statistical Physics* (Oxford: Pergamon) chap 13
Mittag L and Stephen M J 1971 *J. Math. & Phys.* **12** 441

Potts R B 1952a *Proc. Camb. Phil. Soc.* **48** 106
—— 1952b *Phys. Rev.* **88** 352
Priest R G 1971 *Phys. Rev. Lett.* **26** 423
Shultz T D 1971 *Molec. Cryst. Liquid Cryst.* **14** 147
Siegert A J F 1963 *Brandeis Lectures* vol 3 (New York: Benjamin) p 159
Straley J P and Fisher M E 1973 *J. Phys.* A: *Math., Nucl. Gen.* **6** 1310
Wallace D J and Zia R K P 1975 *J. Phys.* C: *Solid St. Phys.* **8** 839
Wegner F J 1972 *Phys. Rev.* B **6** 1891
Wilson K G 1971 *Phys. Rev.* B **4** 3174, 3184
—— 1972 *Phys. Rev. Lett.* **28** 548
Wilson K G and Kogut J 1974 *Phys. Rep.* **12** 75
Zia R K P and Wallace D J 1975 *J. Phys.* A: *Math. Gen.* **8** in the press